Strategic Ambiguity in Games

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22 March 2021

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Literature: Part I

Decision making under Ambiguity

1. Books and surveys

Gilboa, I. (2009). *Decision Theory*.
Kreps, D. (1988). *Notes on the Theory of Choice*.
Wakker, P. (2010). *Prospect Theory for Risk and Ambiguity*.
Machina, M. & Siniscalchi, M. (2013). *Ambiguity and Ambiguity Aversion. [Handbook: Risk and Uncertainty]*

2. Important articles

CEU: Schmeidler, D. (1989).
MEU: Gilboa, I. & Schmeidler, D. (1989).
Recursive model: Segal, U, (1990).
Smooth model: Klibanoff, P., Marinacci, M. & Mukerji, S. (2005).
α-MEU: Ghirardato, P., Marinacci, M. (2002).

Literature: Part I

Decision making under Ambiguity

3. Related approaches
Vector Expected Utility: Siniscalchi, M. (2009).
Variational Preferences: Maccheroni, F., Marinacci, M., Rusticchini, A. (2006).
CEU with neo-additive capacities: Chateauneuf, A., Eichberger, J., Grant, S. (2007).
Confidence functions: Chateauneuf, A., Faro, J.H. (2009).

4. Alternative approaches
Incomplete preferences: Bewley (2002).
Partial information and belief functions: Jaffray (1989).

Decisions under risk and uncertainty

- Objects of the decision are
  - *probability distributions over outcomes (lotteries), or*
  - *state-contingent outcomes.*
- Preferences order
  - a *set of lotteries* or
  - a set of *state-contingent outcomes (acts, actions)*.

It is usual to distinguish

- *decisions under risk:*
  - *probabilities of outcomes* are part of the *information* of the decision maker, i.e.
  - objects of the decision are *lotteries,*
- *decisions under uncertainty:*
  - *probabilities of outcomes are not* part of the *information* of the decision maker, i.e.
  - objects of the decision are *state-contingent outcomes*.

Decisions under uncertainty

- *von Neumann & Morgenstern*
  - *Expected Utility (EU)*
  - choice of *lotteries*
- *Savage*
  - *Subjective Expected Utility (SEU)*
  - choice of *actions*,
    i.e. state-contingent outcomes
- *Anscombe & Aumann*
  - both *subjective* and *objective probabilities*,
  - choice of *actions* with *lotteries as outcomes*,
    i.e., state-contingent lotteries.

Lotteries: von Neumann-Morgenstern approach

- Consider a *finite* set of outcomes:

$$X := \{x_1, ..., x_n\}.$$

- The set of*probabilities over the outcomes* in X is

$$\Delta^n := \{(p_1, ..., p_n) \in \mathbb{R}^n_+ | \sum_{i=1}^{n} p_i = 1\} \subset \mathbb{R}^n.$$

- One assumes that there is a *preference order*

$$\succeq \text{ on } \Delta^n.$$

Proposition 1.1 (*Debreu, 1952*)

The following statements are equivalent:
(i) The preference order $\succeq$ on $\Delta^n \subset \mathbb{R}^n$ satisfies Axioms
A1: Completeness,
A2: Transitivity, *and*
A3: Continuity.
(ii) There exists a function $V : \Delta^n \to \mathbb{R}$ such that for all $p, q \in \Delta^n$,

$$p \succeq q \quad \Leftrightarrow \quad V(p) \geq V(q).$$

Remark 1.1

The general representation over lotteries does not distinguish probabilities and outcomes.

Lotteries: von Neumann-Morgenstern approach

Expected utility preferences

Expected Utility

$$V(p) = \sum_{i=1}^{n} p_i \cdot u(x_i)$$

Properties of EU-preferences

- von Neumann-Morgenstern utilities are unique up to a linear affine transformation: for $b > 0$,

$$v(x) = a + b \cdot u(x).$$

v and u represent the same preferences over Δ^3.

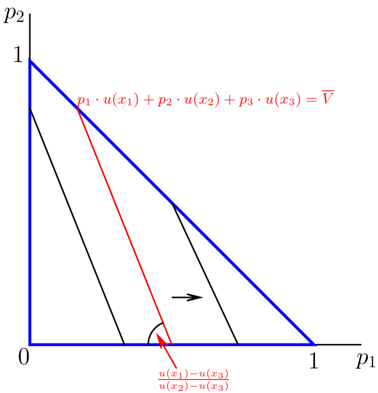
- Hence, one can normalise two values of the von Neumann-Morgenstern utility function: for $u(x^*) > u(x_*)$, define $v(x) := \frac{u(x) - u(x_*)}{u(x^*) - u(x_*)}$, then

$$v(x^*) = 1 \text{ and } v(x_*) = 0.$$

Lotteries: von Neumann-Morgenstern approach

Example 1.1

Indifference curves are linear and parallel



$$p_2 = \frac{\overline{V} - u(x_3)}{u(x_2) - u(x_3)} - \frac{u(x_1) - u(x_3)}{u(x_2) - u(x_3)} \cdot p_1.$$

Risk attitudes

$$L = \langle x_1, x_2; \pi, 1-\pi \rangle$$

- A *risk-averse* decision maker will
  - pay a *positive risk premium P_L* and
  - has a *certainty equivalent* Q_L
    which is *lower* than the expected value of the lottery L,

$$P_L > 0 \iff Q_L < \sum_{i=1}^{n} p_i x_i.$$

Risk attitudes

Risk attitudes and the von Neumann-Morgenstern utility function

Risk attitudes can be characterised by the *curvature* of the von Neumann-Morgenstern utility function $u(\cdot)$:

risk attitude	**curvature of $u(\cdot)$**	**risk premium**
risk averse	concave	positive
risk neutral	linear	null
risk loving	convex	negative

risk averse	$u(\sum_{i=1}^{n} p_i x_i) > \sum_{i=1}^{n} p_i u(x_i))$
risk neutral	$u(\sum_{i=1}^{n} p_i x_i) = \sum_{i=1}^{n} p_i u(x_i))$
risk loving	$u(\sum_{i=1}^{n} p_i x_i) < \sum_{i=1}^{n} p_i u(x_i))$

Lotteries: von Neumann-Morgenstern approach

Expected utility

One seeks axioms (assumptions) which guarantee the existence of an expected utility representation:

$$V(p) := \sum_{i=1}^{n} p_i \cdot u(x_i).$$

Axiom A4: Independence

For all $p, q, r \in \Delta^n$ and all $\alpha \in [0, 1]$,

$$\alpha \cdot p + (1 - \alpha) \cdot r \succeq \alpha \cdot q + (1 - \alpha) \cdot r \quad \Leftrightarrow \quad p \succeq q.$$

Lotteries: von Neumann-Morgenstern approach

Expected utility

Proposition 1.2

The following statements are equivalent:

*(i) The preference order $\succeq$ on $\Delta^n \subset \mathbb{R}^n$ satisfies Axioms **A1, A2, A3**, and **A4.***

(ii) There exists a function $u : X \to \mathbb{R}$ such that for all $p, q \in \Delta^n$,

$$p \succeq q \quad \Leftrightarrow \quad \sum_{i=1}^{n} p_i \cdot u(x_i) \geq \sum_{i=1}^{n} q_i \cdot u(x_i).$$

General lotteries for arbitrary sets of outcomes X

Probability measures

Consider

- a set X and
- an algebra of *events*, i.e., an algebra of subsets of X, $\mathcal{X}$.

A *probability measure* is a function

$$P : \mathcal{X} \to [0, 1]$$

with the following properties

$$
\begin{aligned}
&\text{(a)} \quad P(E) \geq 0 \quad \text{for all } E \in \mathcal{X}, \\
&\text{(b)} \quad P(X) = 1, \\
&\text{(c)} \quad P(E \cup F) = P(E) + P(F) \\
&\qquad\qquad \text{for all } E, F \in \mathcal{X} \text{ such that } E \cap F = \emptyset.
\end{aligned}
$$

Lotteries with arbitrary sets of outcomes X

A special case: general lotteries *(simple probability measures)*

A (general) lottery is a *probability measure* with *finite support*:

- $P(E) = 1$ for a *finite set*
  or, equivalently,
- $\operatorname{supp} P := \{x \in X | \; P(x) > 0\}$ is a *finite set* .

Denote by $\mathcal{P}_S$ the set of *general lotteries* (simple probability measures) on X.

An alternative continuity axiom.

Axiom A3*: Archimedean axiom

For all $P, Q, R \in \mathcal{P}$ with $P \succ Q \succ R$ there exist numbers $\alpha, \beta \in (0, 1)$ such that

$$\alpha \cdot P + (1 - \alpha) \cdot R \succeq Q \succeq \beta \cdot P + (1 - \beta) \cdot R.$$

Expected utility for general lotteries

Proposition 1.3

The following statements are equivalent:
(i) The preference order $\succeq$ *on* $\mathcal{P}_s$ *satisfies*
Axioms ***A1, A2, A3*, and A4.***
(ii) There exists a function $u : X \to \mathbb{R}$ *such that, for all* $P, Q \in \mathcal{P}_s$*,*

$$P \succsim Q \;\; \Leftrightarrow \;\; \sum_x P(x) \cdot u(x) \; \geq \; \sum_x Q(x) \cdot u(x).$$

Lotteries with arbitrary sets of outcomes X

Extensions and remarks

- It is possible to derive the expected utility representation also for preferences $\succsim$ on a set of general probability measures $\mathcal{P}$

$$P \succsim Q \quad \Leftrightarrow \quad \int u(x)\, dP \geq \int u(x)\, dQ.$$

by adding a further axiom (*monotony*).

- The *von Neumann-Morgenstern utility function*

$$u : X \to \mathbb{R}$$

must be *bounded*, i.e., there must exist a number K such that for all $x \in X$,

$$-K \leq u(x) \leq K.$$

This condition is always satisfied if X is a *finite* or a *compact* set.

Choice over acts

State-contingent outcomes (acts)

- states of the world: $s \in S$,
- consequences: $x \in X$,
- acts: $f \in \mathcal{F} = \{g | \, g : S \to X\}$.

Events are subsets of the state space: $E \subset S$.

Example 2.1

$$S = \{s_1, s_2\}, \qquad X = \{A, B, C\},$$

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
s_1	A	A	A	B	B	B	C	C	C
s_2	A	B	C	A	B	C	A	B	C

Choice over acts

Example 2.2 (a special case of acts: *bets*)

Consider an urn with black and white balls. A ball will be drawn randomly.

- *Bet on white:* $\begin{cases} 1 & \text{if a white ball is drawn,} \\ 0 & \text{if a black ball is drawn.} \end{cases}$

- *Bet on black:* $\begin{cases} 1 & \text{if a black ball is drawn,} \\ 0 & \text{if a white ball is drawn.} \end{cases}$

- *states of the world:* $s \in S := \{black, white\}$,
- *consequences:* $x \in X := \{0, 1\}$,
- *acts:* $f \in \mathcal{F} = \{g \mid g : S \to X\}$,
  $e.g.,\ f(white) = 1,\ f(black) = 0.$

Choice over acts

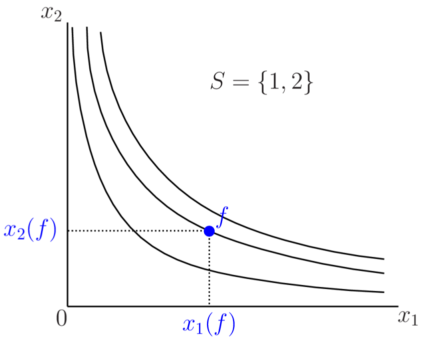
State-contingent outcomes

If the set of states S is *finite*, then one can view a *choice of act* as the *choice of a state-contingent outcome vector*:

$$x(f) = (x_1(f), ..., x_s(f), ..., x_S(f))$$

with

$$x_s(f) := f(s).$$



Savage's approach

Notation

For any two acts $f, g \in \mathcal{F}$ and any event E, the *act* $f_E g$ is defined as

$$f_E g(s) := \begin{cases} f(s) & \text{for} \quad s \in E \\ g(s) & \text{for} \quad s \notin E \end{cases} .$$

Special cases:

- Constant acts x_S :

$$x_S(s) = \quad x \quad \text{for} \quad s \in S.$$

  Abusing notation one writes often x instead of x_S.

- Two-outcome acts $x_E y$:

$$x_E y(s) := \begin{cases} x & \text{for} \quad s \in E \\ y & \text{for} \quad s \notin E \end{cases} .$$

Savage's approach

In order to say when an event E is null, one needs the concept of a preference relation conditional on event E.

Definition 2.1 (*Conditional preferences* $\succsim_E$)

For any $E \subseteq S$ and any acts f, g, h, define the preference relation conditional on event E, $\succsim_E$, by

$$f \succsim_E g \Longleftrightarrow f_E h \succsim g_E h.$$

Definition 2.2 (*null-event*)

An event $E \subseteq S$ is null, if for all acts $f, g \in \mathcal{F}$

$$f \sim_E g.$$

Axiom P1: Weak order

$$\succsim \quad \text{is a} \quad \textit{weak order},$$

i.e., **A1** (completeness) and **A2** (transitivity) hold.

Axiom P2: "Sure-thing principle"

For any four acts $f,\ g,\ h,\ h' \in \mathcal{F}$ and any $E \subset S$

$$f_E h \succsim g_E h \quad \Longleftrightarrow \quad f_E h' \succsim g_E h'.$$

Remark 2.1

*Axioms **P1** and **P2** imply that, for all non-null events $E \subset S$, the conditional preference order $\succsim_E$ satisfies Axiom P1.*

Axiom P3: State independence

For any non-null $E \subset S$,

$$x_E h \succeq_E y_E h \quad \Leftrightarrow \quad x \succeq y.$$

Axiom P4: Outcome independence

For all outcomes $x, y, x', y' \in X$ such that $x \succ y$ and $x' \succ y'$ and all events $A, B \subset S$,

$$x_A y \succeq x_B y \quad \Leftrightarrow \quad x'_A y' \succeq x'_B y'.$$

Axiom P5: Non-trivial choice

There exist $f, g \in \mathcal{F}$ such that

$$f \succ g.$$

Remark 2.2

- ***P3*** *implies that*
  the von Neumann-Morgenstern utility function is state-independent.
- ***P4*** *implies that*
  the probabilities do not depend on the outcomes.
- ***P3*** *and* ***P5*** *imply that*
  S is not a null event.

Axiom P6: Partition of state space

For any acts $f, g, h \in \mathcal{F}$ with $f \succ g$, there exists a *finite partition* of the state space S,

$$\{E_1, E_2, ..., E_n\},$$

such that

$$h_{E_i} f \succ g \text{ and } f \succ h_{E_i} g$$

for all $i = 1, ..., n$.

Axiom P7: Dominance

For any $f, g \in \mathcal{F}$ and any $E \subset S$.

- If $f \succsim_E g(s)$ for all $s \in E$ then $f \succsim_E g$.
- If $g(s) \succsim_E f$ for all $s \in E$ then $g \succsim_E f$.

Remark 2.3

*Axioms **P6** and **P7** guarantee the existence of a unique non-atomic and finitely additive probability measure.*

Savage's approach

Theorem 2.1 (*Savage 1954*)

The following statements are equivalent:
(i) The preference order $\succeq$ on $\mathcal{F}$ satisfies axioms

P1, P2, P3, P4, P5, P6, P7.

(ii) There exist

- *a non-atomic finitely additive probability measure p on S,*
- *a bounded unique (up to an affine transformation) von Neumann-Morgenstern utility function $u : X \to \mathbb{R}$,*

such that

$$f \succsim g \quad \Leftrightarrow \quad \int u(f(s))\, dp(s) \geq \int u(g(s))\, dp(s).$$

Subjective expected utility (SEU)

In order to derive a *Subjective Expected Utility (SEU) representation,*

- *Savage (1954)* allows for *arbitrary states* S and *arbitrary outcomes* X:

$$f : S \to X.$$

  In this case, one needs many axioms because the outcome space X has little structure.
- *Anscombe and Aumann (1964)* consider *general lotteries over outcomes* X as consequences:

$$f : S \to \mathcal{P}_s(X).$$

  In this case, the outcome space $\mathcal{P}_s(X)$ is a set of lotteries and has a nice structure,
  e.g., one can form convex combinations of lotteries $p, q \in \mathcal{P}_s(X)$ for any $\alpha \in [0, 1]$:

$$\alpha p + (1 - \alpha) q \in \mathcal{P}_s(X).$$

Subjective and objective probabilities

- A finite set of states: S,
- a set of consequences X,
- the set of general lotteries on X
  (set of simple probability distributions on X):
  $\mathcal{P}_s(X)$.

An *Anscombe-Aumann act (horse race lottery)* associates a lottery over outcomes with a state,

$$h : S \to \mathcal{P}_s(X).$$

The *set of all Anscombe-Aumann acts* (*horse race lotteries)* is denoted by

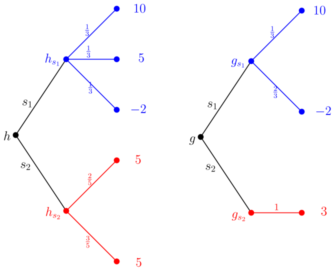
$$\mathcal{H}.$$

Anscombe-Aumann's approach

Horse race lotteries

Example 2.3

- *States:* $S = \{s_1, s_2\}$,
- *Outcomes:* $X = [-10, 10]$
- *Acts:* $h = (h_{s_1}, h_{s_2})$, $g = (g_{s_1}, g_{s_2}) \in \mathcal{H}$.



Anscombe-Aumann's approach

Notation

- The *(objective) probability of outcome* $x \in X$ in state $s \in S$:

$$h_s(x).$$

  The *lottery* in state $s \in S$:

$$h_s := (h_s(x))_{x \in \text{supp}\, h_s}$$

- *Mixture* $h_\alpha g$ *of two Anscombe-Aumann acts* $h, g \in \mathcal{H}$:
  For any $\alpha \in [0, 1]$ the Anscombe-Aumann act
  $h_\alpha g : S \to \mathcal{P}_s(X)$ is defined by

$$(h_\alpha g)_s := \alpha \cdot h_s + (1 - \alpha) \cdot g_s$$

  for each $s \in S$.
- For $h \in \mathcal{H}$ and $p \in \mathcal{P}_s(X)$, (p, h_s) denotes the *Anscombe-Aumann act (horse race lottery)*:

$$(p, h_{-s}) := (h_1, ..., h_{s-1}, p, h_{s+1}, ..., h_S).$$

Axiom A1. Completeness

$$f \succsim g \text{ or } g \succsim f.$$

Axiom A2. Transitivity

$$f \succsim g \text{ and } g \succsim h \text{ imply } f \succsim h.$$

Axiom A3. Independence

For all $h, h', g \in \mathcal{H}$ and all $\alpha \in [0, 1]$,

$$h \succeq h' \Leftrightarrow h_\alpha g \succeq h'_\alpha g.$$

Anscombe-Aumann's approach

Axioms

Axiom A4. Archimedean axiom

For all $h, h', h'' \in \mathcal{H}$ such that $h \succ h' \succ h''$, there are numbers $\alpha, \beta \in (0, 1)$ such that

$$h_\alpha h'' \succ h' \succ h_\beta h''.$$

Axiom A5. Non-trivial preferences

There are $h, h' \in \mathcal{H}$ such that $h \succ h'$.

Axiom A6. State-independent von Neumann-Morgenstern utilities

For $h \in \mathcal{H}$ and $p, q \in \mathcal{P}_s(X)$ such that $(p, h_{-s}) \succ (q, h_{-s})$,

$$(p, h_{-s'}) \succ (q, h_{-s'})$$

for all *non-null* states $s' \in S$.

Anscombe-Aumann's approach

Theorem 2.2 (*Anscombe & Aumann 1963*)

The following statements are equivalent:
(i) The preference order $\succeq$ on $\mathcal{H}$ satisfies axioms ***A1, A2, A3, A4, A5, A6****.*
(ii) There exist
- *a non-constant and unique (up to a positive affine transformation) von Neumann-Morgenstern utility function u on X, and*
- *a unique probability distribution π on S such that*

$$h \succeq g \quad \Leftrightarrow$$

$$\sum_{s \in S} \pi(s) \cdot \left[\sum_{x \in \text{supp}\, h_s} h_s(x) \cdot u(x) \right]$$

$$\geq \sum_{s \in S} \pi(s) \cdot \left[\sum_{y \in \text{supp}\, g_s} g_s(x) \cdot u(x) \right].$$

Paradoxa and experimental observations

- *Allais paradox* ,
- *Ellsberg paradox* ,
- Kahnemann & Tversky (1979): *Prospect Theory*.

- There are two famous experiments
  challenging Expected Utility Theory:
  - *Allais (1953): lotteries,*
  - *Ellsberg (1961)*: *acts*.

- There is a large number of experimental evidence on decision making under uncertainty:
  - *Camerer and Weber (1992),*
  - *Gonzales and Wu (1999),*
  - *Kilka and Weber (2001).*

Allais Paradox

	probabilities		
	0.1	0.01	0.89
Lottery A	1,000,000	1,000,000	1,000,000
Lottery B	5,000,000	0	1,000,000
Lottery C	1,000,000	1,000,000	0
Lottery D	5,000,000	0	0

- Subjects had to choose first between *Lottery A* and *Lottery B,* and then between *Lottery C* and *Lottery D.*
- Most subjects revealed the following preferences:

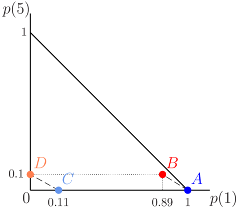
$$A \succ B \text{ and } D \succ C.$$

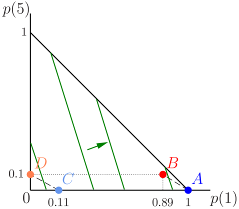
Allais Paradox

Choosing

$$A \succ B \text{ and } D \succ C$$

is *inconsistent with the Expected Utility Hypothesis*.





Allais Paradox

- The framework of the *Allais' paradox* is the *choice over lotteries*, i.e., *objective* probability distributions over outcomes.
- The *Allais' paradox* puts into question the Expected Utility Hypothesis (EU).
- It suggests that the expected utility functional may not always be a good representation of actual preferences over lotteries.
- Quiggin (1982) suggests *Rank-Dependent Expected Utility* as an alternative representation which can deal with preferences of the Allais-paradox type.
- Kahnemann & Tversky (1979) suggest *Prospect Theory*.

Ellsberg paradox

- A second group of paradoxa puts into question whether *people's beliefs* can be represented by *a unique subjective probability distribution* as suggested by the *Subjective Expected Utility Hypothesis* (*SEU*) of Savage (1954).
- Ellsberg (1961) shows by *two examples* that actual behaviour may not be consistent with the assumption of a subjective probability distribution.

Ellsberg Paradox I: two urns

Subjects have to choose between two lotteries and two urns:

	Urn I	
	50	50
	B	*R*
Lottery A	100	0
Lottery B	0	100

	Urn II	
	100	
	B	*R*
Lottery A	100	0
Lottery B	0	100

Suppose you hold *Lottery A*:
Which urn would you prefer to bet on?

Urn I or *Urn II?*

Suppose you hold *Lottery B*:
Which urn would you prefer to bet on?

Urn I or *Urn II?*

Ellsberg Paradox I: two urns

Most subjects chose **Urn I** in both cases.

Such choices are inconsistent with the assumption of *a unique subjective probability distribution* over draws from *Urn II*:

- A *preference for Urn I* given *Lottery A* implies:

$$0.5 = \Pr(B| \quad \text{Urn I} \quad) > \Pr(B| \quad \text{Urn II} \quad).$$

- A *preference for Urn I* given *Lottery B* implies:

$$0.5 = \Pr(R| \quad \text{Urn I} \quad) > \Pr(R| \quad \text{Urn II} \quad).$$

Ellsberg Paradox II: Three-colours urn

	Urn		
	30	60	
	R	*B*	*Y*
Lottery A	100	0	0
Lottery B	0	100	0
Lottery C	100	0	100
Lottery D	0	100	100

Subjects had to choose

- first between

  Lottery A and *Lottery B*

  and
- then between

  Lottery C and *Lottery D*.

Ellsberg Paradox II: Three-colours urn

- Most subjects revealed the following preferences:

$$A \succ B \quad \text{and} \quad D \succ C.$$

- Such behaviour is inconsistent with the assumption of a unique subjective probability distribution over $S = \{R, B, Y\}$:

$$\begin{array}{ccc} A \succ B & \text{implies} & \Pr(R) > \Pr(B) \\ & \text{and} & \\ D \succ C & \text{implies} & \Pr(R) + \Pr(Y) < \Pr(B) + \Pr(Y). \end{array}$$